

Lifetime of Unstable Hairy Black Holes

Shahar Hod

*The Ruppin Academic Center, Emeq Hefer 40250, Israel and
The Hadassah Institute, Jerusalem 91010, Israel*

(Dated: March 5, 2008)

During the last two decades solutions of black holes with various types of “hair” have been discovered. Remarkably, it has been established that many of these hairy black holes are unstable—under small perturbations the hair may collapse. While the static sector of theories admitting hair is well explored by now, our picture of the dynamical process of hair-shedding is still incomplete. In this Letter we provide an important ingredient of the nonlinear dynamics of hair collapse: we derive a universal lower bound on the lifetime of hairy black holes. It is also shown that the amount of hair outside of a black-hole horizon should be fundamentally bounded.

Wheeler’s dictum “a black hole has no hair” [1] has played a major role in the development of black-hole physics [2, 3]. The conjecture implies that black holes are fundamental objects: they should be described by only a few parameters, very much like atoms in quantum mechanics. The fact that stationary black holes are specified by conserved charges which are associated with a Gauss-like law (mass, charge, and angular momentum) was proven explicitly in Einstein vacuum theory and Einstein-Maxwell theory [4]. Early no-hair theorems also excluded scalar [5], massive vector [6], and spinor [7] fields from the exterior of a stationary black hole.

However, later day developments in particle physics have lead to the somewhat surprising discovery of various types of “hairy” black holes, the first of which were the “colored black holes” [8]. These are static black hole solutions of the Einstein-Yang-Mills (EYM) theory that require for their complete specification not only the value of the mass, but also an additional integer (counting the number of nodes of the function characterizing the Yang-Mills field outside the horizon) that is not, however, associated with any conserved charge. Following the discovery of the colored black holes, other hairy solutions have been found [9, 10, 11].

Remarkably, many of these hairy black-hole solutions were proven to be unstable [12, 13, 14]. This implies that under small perturbations the black hole would lose its hair. Numerical studies of hairy black-hole spacetimes [15, 16] have revealed two distinct mechanisms of losing unstable hair: the hair may be dispersed to infinity, leaving the original black hole virtually unchanged, or it may collapse into the black hole, in which case the original black hole gets bigger.

While the *static* sector of theories admitting hair is well understood by now, we still lack a complete picture of the *dynamical* processes by which perturbed, unstable hairy black holes shed their hair. In particular, the fact that many of the hairy black-hole solutions were found to be dynamically unstable promotes the fundamental question: what is the lifetime of an unstable hairy black hole?

In this Letter we derive a universal lower bound on the

dynamical lifetime, τ_{hair} , of perturbed, unstable hairy black-hole solutions. Our derivation is based on standard results of information theory and universal thermodynamic considerations.

A fundamental question in quantum information theory is what is the maximum rate at which information/entropy [17] may be changed by a signal of duration τ and energy $\Delta\mathcal{E}$. The answer to this question is given by the Bekenstein-Bremmermann relation [18, 19, 20]:

$$\frac{\Delta S}{\tau} \leq \pi \Delta\mathcal{E}/\hbar. \quad (1)$$

(We use gravitational units in which $G = c = k_B = 1$.) We shall now use this relation in the context of black-hole dynamics.

Accretion of perturbed, unstable hair into a black hole would increase the black-hole mass [21] by an amount $\Delta\mathcal{E}$ ($\Delta\mathcal{E} \leq \mathcal{M}_{hair}$, where \mathcal{M}_{hair} is the mass of the black-hole hair outside the horizon). As a consequence its area (entropy) would also increase. This implies that information/entropy is actually flowing into the black hole during this dynamical process.

One may now derive an upper bound on the dynamical rate at which the hair collapse (or equivalently, an upper bound on the rate at which the black-hole entropy increases due to hair “swallowing”). For simplicity we focus here on spherically symmetric spacetimes. The mass $m(r_H)$ contained within the black-hole horizon is related to the horizon’s radius by $m(r_H) = \frac{1}{2}r_H$ [3], and the mass of the outside hair is given by $\mathcal{M}_{hair} = \mathcal{M}_{total} - \frac{1}{2}r_H$. It is worth mentioning that [22] provides a detailed discussion about the relation between the bare mass of the black hole (without hair), the mass of the hair, and the ADM mass of the black-hole spacetime.

An increase $\Delta\mathcal{E}$ in the mass contained within the black hole (due to hair swallowing) would increase the black-hole radius from r_H to $r_H + 2\Delta\mathcal{E}$. Taking cognizance of the black-hole area-entropy relation $S_{BH} = A/4\hbar = \pi r_H^2/\hbar$ [2], one finds that the corresponding change in black-hole entropy is given by $\Delta S_{BH} = \pi[(r_H + 2\Delta\mathcal{E})^2 - r_H^2]/\hbar$. Substituting

$$\Delta S_{BH} = 4\pi\Delta\mathcal{E}(r_H + \Delta\mathcal{E})/\hbar, \quad (2)$$

into Eq. (1), we obtain a universal lower bound on the lifetime τ_{hair} (as measured by a static observer at infinity) of a perturbed, unstable hairy black hole:

$$\tau_{hair} \geq 4(r_H + \Delta\mathcal{E}) \geq 4r_H. \quad (3)$$

It is worth noting that Núñez et. al. [3] have proved a very nice theorem according to which the “hairsphere”, the region where the nonlinear behavior of the black-hole hair is present, must extend beyond $\frac{3}{2}r_H$. This would imply that the collapse time of perturbed, unstable hair into the black hole is roughly bounded by $\geq \frac{1}{2}r_H$. The universal bound, Eq. (3), is (at least) an order of magnitude stronger. This dynamical bound, $\tau_{hair} \geq 4(r_H + \Delta\mathcal{E}) \geq 4r_H$, is in fact complimentary to the spatial bound $r_{hair} \geq \frac{3}{2}r_H$ of [3].

We would like to emphasize here that hairy black-hole solutions are characterized by several different length/time scales:

- The radius of the black-hole horizon, r_H .
- The mass of the outside hair, \mathcal{M}_{hair} [23].
- The inverse of the black-hole temperature T_{BH} (proportional to the surface gravity).
- In some hairy solutions there is additional length scale set by the dimensional coupling constant of the theory (for example, the reciprocal of the YM coupling constant, g) [9]. Likewise, in theories with massive hairy fields [11] there is another characteristic scale—the Compton length of the field, \hbar/mass .

The diversity of these independent length/time scales which characterize hairy black-hole solutions, implies that any attempt to obtain a bound on the lifetime of hairy black holes from naive dimensionality considerations would be too presumptuous—there are simply too many independent scales and dimensional parameters in the problem. In principle, the characteristic dynamical timescale could have turned out to be any complicated combination of the above mentioned parameters. However, the analytically derived bound, $\tau_{hair} \geq 4r_H$ [24], is remarkably simple and universal in the sense that it is independent of all other dimensional parameters present in the theory.

Testing the bound.— The early stages of a full, nonlinear collapse of unstable hair can be described by linearized perturbations. Thus the total lifetime of the unstable hairy black hole is bounded from below by the characteristic time which describes the early growth of linear perturbations. (The total lifetime is actually longer and includes the phase of nonlinear dynamics

which follows the linear regime.) The linear instability time is given by the reciprocal of the eigenvalue σ corresponding to the unstable mode of the hairy black hole (a linearized perturbation mode which grows according to $\sim e^{\sigma t}$). One therefore concludes that the total lifetime of an unstable hairy black is bounded by $\tau_{hair} \geq 1/\sigma$. Thus, a sufficient (but not a necessary) condition for the validity of the universal bound, Eq. (3), is that the instability eigenvalue σ is bounded by

$$\sigma \leq \sigma_{max} \equiv [4(r_H + \Delta\mathcal{E})]^{-1} \leq (4r_H)^{-1}. \quad (4)$$

As an example, we display in Table I the numerically computed (see Ref. [25]) eigenvalues σ corresponding to the unstable mode of the canonical EYM (colored) hairy black holes. We also present the ratio σ/σ_{max} . We first consider the weak version of the bound, in which case we take $\Delta\mathcal{E} \rightarrow 0$. One finds $\sigma/\sigma_{max}^{weak} < 1$ for all black holes, in accord with our analytical prediction.

It has been shown [15, 16] that, depending on the initial perturbation most of the unstable hair may collapse into the black hole. We should therefore check the validity of our bound in its strong version, in which case we take $\Delta\mathcal{E} = \mathcal{M}_{hair}$. We display the ratio $\sigma/\sigma_{max}^{strong}$ in Table I, from which one learns that the unstable hairy black holes conform to the bound Eq. (4) even in its strong form. (In fact, the colored black holes are actually very close of saturating this dynamical bound.) It is worth mentioning that the EYM solutions have a second type of instability (“sphaleron” or topological one), which may develop outside the magnetic ansatz to which the solution itself belongs, see [26, 27]. These two types of instabilities have comparable spectrums and are both relevant for the lifetimes of the EYM hairy black holes.

r_H	\mathcal{M}_{hair}	σ	$\sigma/\sigma_{max}^{weak}$	$\sigma/\sigma_{max}^{strong}$
0.0	0.8286	0.2292	0.000	0.759
0.4	0.6589	0.2071	0.331	0.877
0.6	0.5791	0.1936	0.465	0.913
1.0	0.4369	0.1637	0.655	0.941
2.0	0.2349	0.1067	0.854	0.954
3.0	0.1571	0.0744	0.893	0.940
4.0	0.1168	0.0562	0.899	0.925
5.0	0.0923	0.0447	0.894	0.911

TABLE I: Ratio between the eigenvalue σ corresponding to the unstable mode of the colored hairy black holes and σ_{max} , as defined by the bound Eq. (4). Here $\sigma_{max}^{weak} \equiv (4r_H)^{-1}$ and $\sigma_{max}^{strong} \equiv [4(r_H + \mathcal{M})_{hair}]^{-1}$. One finds $\sigma/\sigma_{max} < 1$ for all values of the black-hole radius r_H , in accord with the analytical prediction, Eq. (4).

Discussion and applications.— The universal laws of thermodynamics and information theory provide important insights on the dynamics of unstable hairy black holes. In particular, we have derived a simple lower

bound on the lifetime of perturbed, unstable hair: *If a black hole has hair, then its lifetime cannot be shorter than 4 times the horizon radius.* This temporal bound, $\tau_{\text{hair}} \geq 4(r_H + \mathcal{M}_{\text{hair}}) \geq 4r_H$ is complimentary to the spatial bound $r_{\text{hair}} \geq \frac{3}{2}r_H$ of [3]. It would be highly interesting to verify directly the validity of the analytic bound, Eq. (3), with full nonlinear dynamical calculations.

It is worth pointing out that, although our starting point, Eq. (1) has clear quantum origins, the analytically derived bound on the dynamical lifetime of hairy black holes, Eq. (3), is of pure classical nature. [The \hbar that appears in (1) is canceled out by the \hbar in the black-hole area-entropy relation, $S_{BH} = A/4\hbar$.] This strongly suggests that one should be able to obtain a bound on the lifetime of an unstable black hole by using purely classical arguments. One such (analytic) example is given in Appendix A, where we consider the specific case of the SU(2) Reissner-Nordström black hole. However, such approach would be parameter and model dependent. One would be forced to go into details about the system, and to solve the problem case-by-case for each and every theory in which black-hole hair has been discovered. On the other hand, the inequality (1) is remarkably robust—it therefore enables a simple derivation of a universal, model independent bound like (3).

If the central black hole is small, then the dynamics of its outside fields would hardly be affected by its presence. In such cases, the dimensional parameters of the matter fields (e.g., the dimensional coupling constant, g , of the YM theory) would determine the characteristic dynamical timescale of the outside hair. It is interesting to note that if we take this natural expectation and combine it with the temporal bound $\tau \geq 4(r_H + \mathcal{M}_{\text{hair}})$, we obtain an upper bound for the mass of the black-hole hair outside the horizon. Namely, substituting $\tau \sim g^{-1}$ for the characteristic dynamical timescale would yield,

$$\mathcal{M}_{\text{hair}} \lesssim 1/g - r_H \quad ; \quad r_H \lesssim 1/g. \quad (5)$$

This bound implies that a black hole cannot support an unbounded amount of hair. The mass of the outside hair is fundamentally bounded by the reciprocal of the coupling constant of the theory. Moreover, the amount of allowed hair decreases with increasing black-hole radius. These analytical conclusions are in accord with available numerical data, see e.g., [25]. The transition from hairy black-hole solutions which are dominated by their outside hair ($\mathcal{M}_{\text{hair}} \gg r_H$), to solutions dominated by the central black hole ($\mathcal{M}_{\text{hair}} \ll r_H$) occurs at $r_H \sim 1/g$.

Likewise, in theories with massive hairy fields there is a natural length/time scale in the problem set by the Compton length of the field, \hbar/m , where m is the mass parameter. This scale would characterize the dynamics of the outside fields if the central black hole is small.

Taking $\tau \sim \hbar/m$ in the relation $\tau \geq 4(r_H + \mathcal{M}_{\text{hair}})$, one obtains an upper bound on the mass of the outside hair,

$$\mathcal{M}_{\text{hair}} \lesssim \hbar/m - r_H \quad ; \quad r_H \lesssim \hbar/m. \quad (6)$$

This analytical bound is in accord with known numerical data [3, 11], and implies that $\mathcal{M}_{\text{hair}} \rightarrow 0$ as $r_H \simeq \hbar/m$.

The above upper bounds do not rule out the existence of hairy black-hole solutions. However, they put a severe restriction on the amount of outside hair that a black hole can support. Such a restrictive upper bound can therefore be taken as a more modest alternative to the original no hair conjecture.

Finally, we would like to point out that our approach sheds some new light on the black-hole critical phenomena observed in full nonlinear gravitational collapse [30]. It has been found [15, 16] that unstable hairy black holes may serve as intermediate attractors during a dynamical gravitational collapse. Near-critical evolutions approach the solution of a static hairy black hole, and remain in its vicinity for some amount of time T . The evolution then peels off the static hairy black hole solution, with the remaining field (outside of the horizon) either dispersing to infinity or collapsing and adding a finite amount of mass to the already present black hole [15, 16]. The lingering time, T , in the vicinity of the intermediate attractor (the unstable hairy black-hole solution) is related directly to the reciprocal of the instability eigenvalue σ [25]. Our results therefore set an analytic lower bound on the lifetime, T , of near-critical solutions in Choptuik's nonlinear critical phenomena.

Appendix A: Instability of the SU(2) Reissner-Nordström black hole

In this appendix we consider the linearized stability analysis of the SU(2) Reissner-Nordström (RN) black hole with unit magnetic charge [28]. (This black-hole solution is not regarded as hairy since its outside mass is related to the global magnetic charge. Nevertheless, we present here the stability analysis of this black hole because it has the advantage of being tractable analytically.)

The mass function of the SU(2) RN black hole of asymptotic mass M is given by

$$m(r) = M - \frac{1}{2r}. \quad (\text{A1})$$

This implies that the black-hole horizon is located at $r_H = M + (M^2 - 1)^{1/2}$.

The evolution of linearized perturbations $\xi(r)e^{\sigma t}$ is governed by a Schrödinger-like equation [29]

$$\left(-\frac{d^2}{dx^2} + U(x)\right)\xi = -\sigma^2\xi, \quad (\text{A2})$$

where $dx/dr = [1 - 2m(r)/r]^{-1}$ (this implies $x \rightarrow -\infty$ for $r \rightarrow r_H$, and $x \rightarrow \infty$ for $r \rightarrow \infty$), and the effective potential is given by

$$U(x) = -\frac{1}{r^2}\left(1 - \frac{2m(r)}{r}\right). \quad (\text{A3})$$

Eq. (A2) has at least one bound state, because the effective potential $U(x)$ is everywhere negative and vanishes for $|x| \rightarrow \infty$. Moreover, an upper bound on σ^2 is given by the (minus) minimum of the potential U [29]. Namely, $\sigma \leq \sqrt{|U_{\min}(M)|}$. From here one finds that σ is bounded according to

$$\sigma(M) < [4\beta(M)r_H]^{-1}, \quad (\text{A4})$$

where $\beta(M)$ is a monotonic decreasing function from $\beta(M=1) = 1$ to $\beta(M \rightarrow \infty) = \sqrt{27}/8$. This yields a lower bound on the black-hole lifetime, $\tau \geq 1/\sigma$ which is a bit weaker than our universal bound, Eq. (3).

ACKNOWLEDGMENTS

This research is supported by the Meltzer Science Foundation. I thank Piotr Bizoń for helpful correspondence. I also thank Uri Keshet and Liran Shimshi for stimulating discussions.

Session of Ecole d'été de Physique Theorique, edited by C. De Witt and B. S. De Witt (Gordon and Breach, New York, 1973).

- [2] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); Phys. Today **33**, 24 (1980); Phys. Rev. D **51**, R6608 (1995).
- [3] D. Núñez, H. Quevedo, and D. Sudarsky, Phys. Rev. Lett. **76**, 571 (1996).
- [4] See references in R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [5] J. E. Chase, Commun. Math. Phys. **19**, 276 (1970); J. D. Bekenstein, Phys. Rev. Lett. **28**, 452 (1972); C. Teitelboim, Lett. Nuovo Cimento **3**, 326 (1972).
- [6] J. D. Bekenstein, Phys. Rev. D **5**, 1239 (1972); **5**, 2403 (1972); M. Heusler, J. Math. Phys. **33**, 3497 (1992); D. Sudarsky, Class. Quantum Grav. **12**, 579 (1995).
- [7] J. Hartle, Phys. Rev. D **3**, 2938 (1971); C. Teitelboim, Lett. Nuovo Cimento **3**, 397 (1972).
- [8] P. Bizoń, Phys. Rev. Lett. **64**, 2844 (1990); M. S. Volkov and D. V. Gal'tsov, Sov. J. Nucl. Phys. **51**, 1171 (1990); H. P. Kuenzle and A. K. M. Masood-ul-Alam, J. Math. Phys. **31**, 928 (1990).
- [9] G. Lavrelashvili and D. Maison, Nucl. Phys. B **410**, 407 (1993).
- [10] P. Bizoń and T. Chamj, Phys. Lett. B **297**, 55 (1992); M. Heusler, S. Droz, and N. Straumann, Phys. Lett. B **268**, 371 (1991); **271**, 61 (1991); **258**, 21 (1992).
- [11] B. R. Greene, S. D. Mathur, and C. O'Neill, Phys. Rev. D **47**, 2242 (1993); T. Torii, K. Maeda, and T. Tachizawa, Phys. Rev. D **51**, 1510 (1995).
- [12] N. Straumann and Z. -H. Zhou, Phys. Lett. B **243**, 33 (1990).
- [13] P. Bizoń and R. M. Wald, Phys. Lett. B **267**, 173 (1991).
- [14] M. S. Volkov and D. V. Gal'tsov, Phys. Rept. **319**, 1 (1999).
- [15] M. W. Choptuik, T. Chmaj, and P. Bizoń, Phys. Rev. Lett. **77**, 424 (1996).
- [16] M. W. Choptuik, E. W. Hirschmann, and R. L. Marsa, Phys. Rev. D **60**, 124011 (1999).
- [17] There is a complementary relation between entropy and information: entropy as a measure of one's uncertainty or lack of information about the actual internal state of the system, see e.g: C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication* (Univ. of Illinois Press, Urbana, 1949); S. Goldman, *Information Theory* (Prentice-Hall, New York, 1953); L. Brillouin, *Science and Information Theory* (Academic, New York, 1956).
- [18] J. D. Bekenstein, Phys. Rev. Lett. **46**, 623 (1981).
- [19] H. J. Bremermann, in *Proceedings of Fifth Berkeley Symposium on Mathematical Statistics and Probability*, edited by L. M. LeCam and J. Neyman (Univ. of California Press, Berkeley, 1967).
- [20] We note that there is an ongoing controversy in the literature around the Bekenstein entropy bound and its relevance for the validity of the generalized second law of thermodynamics in the context of black-hole physics, see W. G. Unruh and R. M. Wald, Phys. Rev. D **25**, 942 (1982); Phys. Rev. D **27**, 227 (1983); M. A. Pelath, R. M. Wald, Phys. Rev. D **60**, 104009 (1999); J. D. Bekenstein, Phys. Rev. D **60**, 124010 (1999).
- [21] We assume the validity of the weak energy condition, which insures the positivity of the hair's mass.
- [22] A. Ashtekar, A. Corichi and D. Sudarsky, Classical and Quantum Gravity **18**, 919 (2001).
- [23] We emphasize that the mass of the black-hole hair,

[1] R. Ruffini and J. A. Wheeler, Phys. Today **24** (1), 30 (1971); B. Carter, in *Black Holes*, Proceedings of 1972

- \mathcal{M}_{hair} does not scale linearly with its radius, r_H , see e.g. [25]. These two quantities therefore determine different length/time scales.
- [24] The bound can be made stronger, $\tau_{hair} \geq 4(r_H + \mathcal{M}_{hair})$, as discussed below.
 - [25] P. Bizoń and T. Chmaj, Phys. Rev. D **61**, 067501 (2000).
 - [26] G. V. Lavrelashvili and D. Maison, Phys. Lett. B **343**, 214 (1995).
 - [27] M. S. Volkov, O. Brodbeck, G. V. Lavrelashvili and N. Straumann, Phys. Lett. B **349**, 438 (1995).
 - [28] P. B. Yasskin, Phys. Rev. D **12**, 108 (1975).
 - [29] P. Bizoń, Phys. Lett B **259**, 53 (1991).
 - [30] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).